PRECONDITIONING OPERATORS ON UNSTRUCTURED GRIDS *

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Abstract

We consider systems of mesh equations that approximate elliptic boundary value problems on arbitrary (unstructured) quasi-uniform triangulations and propose a method for constructing optimal preconditioning operators. The method is based upon two approaches: (1) the fictitious space method, i.e. the reduction of the original problem to a problem in an auxiliary (fictitious) space, and (2) the multilevel decomposition method, i.e. the construction of preconditioners by decomposing functions on hierarchical meshes. The convergence rate of the corresponding iterative process with the preconditioner obtained is independent of the mesh size. The preconditioner has an optimal computational cost: the number of arithmetic operations required for its implementation is proportional to the number of unknowns in the problem. The construction of the preconditioning operators for three dimensional problems can be done in the same way.

1 INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be a domain with a piecewise smooth boundary Γ which belongs to the class C^2 and satisfies the Lipschitz condition [21]. In the domain Ω

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we consider the boundary value problem

$$
\begin{aligned}
-\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\, a_{ij}(x)\, \frac{\partial u}{\partial x_j}\, + a_0(x)u = f(x)\,, \qquad x \in \Omega \\
u(x) = 0\,, \qquad x \in \Gamma_0 \\
\frac{\partial u}{\partial N} + \sigma(x)u = 0\,, \qquad x \in \Gamma_1
\end{aligned}
\tag{1.1}
$$

where

$$\frac{\partial u}{\partial N} = \sum_{i,j=1}^{2} a_{i,j}(x)\, \frac{\partial u}{\partial x_j}\, \cos(n, x_i)$$

is the conormal derivative, n denotes the outward normal to Γ, and Γ_0 is a union of a finite number of curvilinear segments, $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 = \bar{\Gamma}_0$. Here $\bar{\Gamma}_0$ denotes the closure of Γ_0.

By $H^1(\Omega, \Gamma_0)$ we denote the subspace of the Sobolev space $H^1(\Omega)$

$$H^1(\Omega, \Gamma_0) = \{v \in H^1(\Omega) \mid v(x) = 0,\ x \in \Gamma_0\}\,.$$

We introduce a bilinear form $a(u, v)$ and a linear functional $l(v)$ as follows:

$$a(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij}(x)\, \frac{\partial u}{\partial x_j}\, \frac{\partial v}{\partial x_i}\, + a_0(x)uv \Big)\, \mathrm{d}x + \int_{\Gamma_1} \sigma(x)uv\, \mathrm{d}x$$

$$l(v) = \int_\Omega f(x)v\, \mathrm{d}x\,.$$

Let us suppose that the operator coefficients and the right-hand side of problem (1.1) are such that the bilinear form $a(u, v)$ is symmetric, elliptic and continuous on $H^1(\Omega, \Gamma_0) \times H^1(\Omega, \Gamma_0)$, i.e.

$$a(u, v) = a(v, u) \qquad \forall u, v \in H^1(\Omega, \Gamma_0)$$

$$\alpha_0 \|u\|^2_{H^1(\Omega)} \le a(u, u) \le \alpha_1 \|u\|^2_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega, \Gamma_0)$$

and the linear functional $l(v)$ is continuous on $H^1(\Omega, \Gamma_0)$:

$$|l(u)| \le \alpha \|u\|_{H^1(\Omega)} \qquad \forall u \in H^1(\Omega, \Gamma_0)\,.$$

The generalized solution $u \in H^1(\Omega, \Gamma_0)$ of problem (1.1) is, by definition, a solution to the projection problem [2]

$$u \in H^1(\Omega, \Gamma_0):\ a(u,v) = l(v) \qquad \forall v \in H^1(\Omega, \Gamma_0)\,. \tag{1.2}$$

It is familiar that under these assumptions concerning $a(u,v)$ and $l(v)$ there exists a unique solution of problem (1.2).

Let a positive parameter h be fixed (we always suppose that h is sufficiently small). Let

$$\Omega^h = \bigcup_{i=1}^{M} \tau_i$$

be a triangulation of the domain Ω (Ω^h is assumed to be a closed set). We suppose that Ω^h is a quasi-uniform triangulation [8], i.e. there exist positive constants l_1, l_2 and s which are independent of h and such that

$$l_1 h \le r_i \le l_2 h\,, \qquad \frac{r_i}{\rho_i} \le s\,, \qquad i = 1, \ldots, M$$

where r_i and ρ_i are radii of circumscribed and inscribed circles for the triangle τ_i, respectively. We also assume that the triangulation boundary Γ^h approximates Γ with an error $O(h^2)$. If $\Gamma_1 = \Gamma$, we suppose that $\Omega \subset \Omega^h$; if $\Gamma_0 = \Gamma$, we suppose that $\Omega^h \subset \Omega$. If $\Gamma_0 \ne \emptyset$ and $\Gamma_1 \ne \emptyset$, we make the following assumption: points where the boundary condition changes should be at triangulation nodes, $\Gamma_1 \subset \Omega^h$ and $\Gamma_0 \subset \overline{(\mathrm{IR}^2 \setminus \Omega^h)}$. Part of Γ^h approximating Γ_0 will be denoted by Γ_0^h, and that for Γ_1 by Γ_1^h. For the triangulation Ω^h, we define the space $H_h(\Omega^h)$ of real continuous functions which are linear on each triangle of Ω^h and vanish at Γ_0^h. We extend these functions on $\Omega \setminus \Omega^h$ by zero.

The solution of the projection problem

$$u^h \in H_h(\Omega^h):\ a(u^h, v^h) = l(v^h) \qquad \forall v^h \in H_h(\Omega^h) \tag{1.3}$$

will be called an approximate solution of problem (1.2). Aspects of approximation of (1.2) by (1.3) have been thoroughly studied (see [8, 17]); we do not consider them here. Each function $u^h \in H_h(\Omega^h)$ is put in standard correspondence with a real column vector $u \in \mathrm{IR}^N$ whose components are values of the function u^h at the corresponding nodes of the triangulation Ω^h. Then

(1.3) is equivalent to the system of mesh equations

$$Au = f$$

$$(Au, v) = a(u^h, v^h) \qquad \forall u^h, v^h \in H_h(\Omega^h) \tag{1.4}$$

$$(f, v) = l(v^h) \qquad \forall v^h \in H_h(\Omega^h)$$

where u^h and v^h are the respective prolongations of vectors u and v; (f, v) is the Euclidean scalar product in $\mathbb{R}^N$.

The main goal of this work is to construct a symmetric positive definite preconditioning operator B for problem (1.4) so as to satisfy the inequalities

$$c_1(Bu, u) \leq (Au, u) \leq c_2(Bu, u) \qquad \forall u \in \mathbb{R}^N \tag{1.5}$$

where positive constants c_1 and c_2 are independent of h; the multiplication of a vector by B^{-1} should be easy to implement.

The preconditioner B is constructed by using the method of fictitious space [13] in two stages. At the first stage, we pass from an arbitrary unstructured triangulation Ω^h to an auxiliary structured non-hierarchical mesh, and at the second stage to a hierarchical mesh (a square mesh on a square containing the original domain Ω). Note that the passage from an arbitrary triangulation to a structured mesh was earlier used in [14]. This paper includes some development of [16] for the case of locally refined grids. Other techniques for constructing the preconditioners on unstructured meshes were proposed in [3, 4, 7, 11, 12, 13, 20]. The construction of preconditioning operators on non-hierarchical grids was considered in [9].

2 REDUCTION TO A STRUCTURED MESH

The preconditioning operator B in (1.5) is constructed on the basis of the lemma of fictitious space [14]. For convenience, we give this lemma here.

Lemma 2.1. *Let H_0 and H be Hilbert spaces with the scalar products $(u_0, v_0)_{H_0}$ and $(u,v)_H$, respectively. Let A_0 and A be symmetric positive definite continuous operators in the spaces H_0 and H :*

$$A_0\colon\ H_0 \to H_0\,, \qquad A\colon\ H \to H\,.$$

Suppose that R is a linear operator such that

$$R\colon\ H \to H_0$$

$$(A_0Rv, Rv)_{H_0} \le c_R(Av, v)_H \qquad \forall v \in H$$

and there exists an operator T such that

$$T\colon\ H_0 \to H\,, \qquad RTu_0 = u_0$$

$$c_T(ATu_0, Tu_0)_H \le (A_0u_0, u_0)_{H_0} \qquad \forall u_0 \in H_0$$

where c_R and c_T are positive constants. Then

$$c_T(A_0^{-1}u_0, u_0)_{H_0} \le (RA^{-1}R^*u_0, u_0)_{H_0} \le c_R(A_0^{-1}u_0, u_0)_{H_0} \qquad \forall u_0 \in H_0\,.$$

The operator R^* is adjoint to R with respect to the scalar products $(u_0, v_0)_{H_0}$ and $(u,v)_H$:

$$R^*\colon\ H \to H_0$$

$$(R^*u_0, v)_H = (u_0, Rv)_{H_0}\,.$$

Note that for constructing and implementing the preconditioner, i.e. the operator $RA^{-1}R^*$, we only require the existence of the operator T. In our case, the role of the operator A_0 is played by A of (1.4), and the role of the space H_0 by $H_h(\Omega^h)$. In order to use Lemma 2.1, we construct a fictitious (auxiliary) space and the corresponding operators. To do this, we embed the domain Ω in a square Π. Let K_i denote the union of triangles in the triangulation Ω^h which have a common vertex z_i, and let d_i be the maximum radius of circle inscribed in K_i. In the square Π, we introduce an auxiliary grid Π^h with a step size $\bar{h}$ such that

$$\bar{h} < \frac{1}{2\sqrt{2}}\ \min_i\ d_i\,. \tag{2.1}$$

Let us assume that $\bar{h} = l \cdot 2^{-J}$, where l is the length of sides of Π and J is a positive integer. We denote the nodes of the grid Π^h by Z_{ij},

$$Z_{ij} = (x_i, y_j)\,, \qquad i,\, j = 0, 1, \ldots, L$$

and the cells of Π^h by D_{ij},

$$D_{ij} = \{(x, y) \mid x_i \le x < x_{i+1},\ y_j \le y < y_{j+1}\}$$

$$\Pi^h = \bigcup_{i,j=0}^{L} D_{ij}\,.$$

Let Q^h denote the minimum figure that consists of cells D_{ij} and contains Ω^h: $\Omega^h \subset Q^h$; let S^h be the set of boundary nodes of Q^h. We subdivide the set S^h into two subsets S_0^h and S_1^h as follows: if

$$\bar{D}_{ij} \cap \Gamma_0 \ne \emptyset$$

all nodes of $D_{ij} \cap S^h$ are in S_0^h

$$S_1^h = S^h \setminus S_0^h\,.$$

Using cell diagonals, we triangulate Q^h and Π^h; hereafter, the designations Q^h and Π^h refer to triangulations as well. Let $H_h(Q^h)$ be the space of real continuous functions which are linear on the triangles of Q^h and vanish at the nodes of S_0^h. It is the space $H_h(Q^h)$ that will be used as the fictitious space in Lemma 2.1.

We now define the projection operator R

$$R\colon\ H_h(Q^h) \to H_h(\Omega^h)$$

the extension operator T

$$T\colon\ H_h(\Omega^h) \to H_h(Q^h)$$

and an easily invertible operator in the space $H_h(\Omega^h)$.

Let us begin with the operator R. For a given mesh function

$$U^h(Z_{ij}) \in H_h(Q^h)$$

we define a function $u^h \in H_h(\Omega^h)$ as follows. Let z_l be a vertex in the triangulation Ω^h; assume that $z_l \in D_{ij}$. We put

$$u^h(z_l) = (TU^h)(z_l) = U^h(Z_{ij})\,. \tag{2.2}$$

The function u^h is equal to zero at nodes $z_l \in \Gamma_0^h$.

Then, let us define the operator T. For a given function $u^h \in H_h(\Omega^h)$, we define a function $U \in H_h(Q^h)$. The function U^h is equal to zero at nodes $Z_{ij} \in S_0^h$. At the other nodes, U is defined as follows. If a cell D_{ij} contains a certain vertex z_l of the triangulation Ω^h, we put

$$U^h(Z_{ij}) = (Tu^h)(Z_{ij}) = u^h(z_l)\,.$$

For each of the remaining nodes $Z_{ij} \in Q^h$, we find the closest vertex z_l of the triangulation Ω^h (if there are several closest vertices, we can choose any of them) and put

$$U^h(Z_{ij}) = (Tu^h)(Z_{ij}) = u^h(z_l)\,.$$

Finally, in the space $H_h(Q^h)$ we define the operator A_Q:

$$(A_Q U, V) = \int_{Q_h} ((\nabla U^h, \nabla V^h) + U^h \cdot V^h)\, dx\, dy \qquad \forall U^h, V^h \in H_h(Q^h). \tag{2.3}$$

where U^h and V^h are the respective prolongations of the vectors U and V.

Theorem 2.1. *There exist positive constants c_3 and c_4, independent of h, such that*

$$c_3(A^{-1}u, u) \leq (RA_Q^{-1}R^* u, u) \leq c_4(A^{-1}u, u) \qquad \forall u \in \mathrm{I\!R}^N\,.$$

Here A, R and A_Q are operators of (1.4), (2.2) *and* (2.3), *respectively; R^* is the transpose of R (we hereafter use the same designation for an operator and its matrix representation).*

Proof. The theorem easily follows from Lemma 2.1, condition (2.1) and the familiar equivalence of H^1-norms of finite-element functions in the spaces $H_h(\Omega^h)$, $H_h(Q^h)$ and the difference counterparts of these norms [17].

Remark 2.1. The implementation of the operator R is equivalent to the piecewise constant interpolation. It is easily seen that the number of arithmetic operations required for multiplying R or R^* by a vector is proportional to the number of nodes in the mesh domain.

Thus, the construction of a preconditioning operator on an unstructured triangulation is reduced to the construction of a preconditioning operator for A_Q. The latter problem is considered in Section 3.

3 FICTITIOUS SPACE AND MULTILEVEL DECOMPOSITION METHODS

In order to find a preconditioning operator for A_Q, we again use Lemma 2.1. Here the fictitious (auxiliary) space is $H_h(\Pi^h)$ which consists of piecewise linear continuous functions vanishing on the boundary $\partial\Pi$ of the square Π. Efficient preconditioning operators in $H_h(\Pi^h)$ are well known; in particular, we may use the BPX preconditioner [6]. To do so, we use the following construction.

We divide the domain $\Pi \setminus \overline{\Omega}$ into two non-intersecting subdomains such that

$$
\begin{aligned}
\Pi\setminus = \bar{\Gamma}_0 \cup \bar{\Gamma}_1\,, \qquad & G_0 \cap G_1 = \emptyset \\
\partial G_0 \cap \partial\Omega = \Gamma_0\,, \qquad & \partial G_1 \cap \partial\Omega = \bar{\Gamma}_1\,.
\end{aligned}
\tag{3.1}
$$

According to (3.1), we represent the triangulation $\Pi^h \setminus Q^h$ as a union of two non-overlapping parts:

$$\overline{\Pi^h \setminus Q^h} = G_0^h \cup G_1^h$$

where G_0^h and G_1^h are mesh approximations of the domains G_0 and G_1, respectively. Further, we denote

$$G = \Omega \cup \Gamma_1 \cup G_1\,, \qquad G^h = Q^h \cup G_1^h$$

$H_h(G^h)$ finite-element space of functions vanishing on ∂G^h. We consider in Π^h the sequence of grids

$$\Pi_0^h, \Pi_1^h, \ldots, \Pi_J^h \equiv \Pi^h$$

with step sizes

$$h_0 = l\,, \quad h_1 = l \cdot 2^{-1}\,, \ \ldots\,, \ h_J \equiv \bar{h} = l \cdot 2^{-J}\,.$$

We triangulate these grids and consider the corresponding finite-element spaces

$$W_0^h \subset W_1^h \subset \ldots \subset W_J^h \equiv H_h(\Pi^h)\,.$$

By $\{\Phi_i^{(l)}\}_{i=1}^{N_l}$ we denote the nodal basis of the space W_l^h, $l = 0, 1, \ldots, J$.

First, let us examine the case of $\Gamma_1 = \Gamma$; accordingly, here $S_1^h = S^h$. By $\tilde{\Phi}_i^{(l)}$ we denote the restriction of the basic function $\Phi_i^{(l)}$ onto Q^h. We put each function $U^h \in H_h(Q^h)$ in correspondence with a function $\tilde{U}^h \in H_h(\Pi^h)$:

$$\tilde{U}^h(Z_{ij}) = \begin{cases} U^h(Z_{ij})\,, & Z_{ij} \in Q^h \\ 0\,, & Z_{ij} \in \Pi^h \setminus Q^h\,. \end{cases}$$

Define

$$C_N^{-1} U^h = \sum_{l=0}^{J} \sum_{\operatorname{supp} \Phi_i^{(l)} \cap Q^h \neq \emptyset} (\tilde{U}^h, \Phi_i^{(l)})_{L_2(\Pi)}\, \tilde{\Phi}_i^{(l)} \qquad \forall U^h \in H_h(Q^h).$$

Theorem 3.1. *There exist positive constants c_5 and c_6, independent of h, such that*

$$c_5(A^{-1}u, u) \le (R C_N^{-1} R^* u, u) \le c_6(A^{-1}u, u) \qquad \forall u \in \mathbb{R}^N\,.$$

Proof. Let us define

$$R_N\colon\ H_h(\Pi^h) \to H_h(Q^h)$$

to be an operator of restriction on Q^h:

$$(R_N U^h)(Z_{ij}) = U^h(Z_{ij}) \qquad \forall Z_{ij} \in Q^h\,.$$

If we subdivide the nodes of Π^h into two groups: (1) the nodes of Q^h (including those of S^h), and (2) the remaining nodes, then we obtain the following matrix representation for R_N (see also [1]):

$$R_N = (IO)$$

where I is the identity matrix corresponding to nodes of group (1), and O is the zero matrix corresponding to nodes of group (2). It is evident that

$$\|R_N U^h\|_{H^1(Q^h)} \leq \|U^h\|_{H^1(\Pi^h)} \qquad \forall U^h \in H_h(\Pi^h)\,.$$

By the theorem of extension of mesh functions [6], there exists the extension operator

$$T_N\colon\ H_h(Q^h) \to H_h(\Pi^h)$$

uniformly bounded with respect to h.

According to Lemma 2.1 and [6], there exist positive constants c_7 and c_8, independent of h, such that

$$c_7(A_Q^{-1}U,U) \leq (R_N C_\Pi^{-1} R_N^* U,U) \leq c_8(A_Q^{-1}u,u) \qquad \forall U$$

where A_Q is the operator of (2.3) and the definition of C_Π^{-1} is

$$C_\Pi^{-1}U^h = \sum_{l=0}^{J}\sum_{i=1}^{N_l}(U^h,\Phi_i^{(l)})_{L_2(\Pi)}\,\Phi_i^{(l)} \qquad \forall U^h \in H_h(\Pi^h)\,.$$

Taking into account the explicit form of R_N, we complete the proof of Theorem 3.1.

Then, let us examine the case of the Dirichlet problem, i.e. $\Gamma_0 = \Gamma$ and, accordingly, $S_0^h = S^h$. We define the preconditioner as follows:

$$C_D^{-1}U^h = \sum_{l=0}^{J}\ \sum_{\operatorname{supp}\Phi_i^{(l)}\subset Q^h}(U^h,\Phi_i^{(l)})_{L_2(Q^h)}\,\Phi_i^{(l)} \qquad \forall U^h \in H_h(Q^h).$$

Theorem 3.2. *There exist positive constants c_9 and c_{10}, independent of h, such that*

$$c_9(A^{-1}u,u) \leq (RC_D^{-1}R^*u,u) \leq c_{10}(A^{-1}u,u) \qquad \forall u \in \mathrm{I\!R}^N\,.$$

Proof. In this case, the equivalence of the operators A_Q and C_D easily follows from the multilevel technique [5, 6, 18, 19] and can be done, for instance, by using of quasi-interpolants from [15]. Then, from Theorem 2.1 we get the assertion of Theorem 3.2.

Finally, we examine the case of mixed boundary conditions, i.e. $\Gamma_0 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$. We denote

$$C_M^{-1} U^h = \sum_{l=0}^{J} \sum_{\substack{\text{supp}\, \Phi_i^{(l)} \subset \, G^h, \\ \text{supp}\, \Phi_i^{(l)} \cap \, Q^h \neq \emptyset}} (\tilde{U}^h, \Phi_i^{(l)})\, \tilde{\Phi}_i^{(l)} \qquad \forall U^h \in H_h(Q^h).$$

Theorem 3.3. *There exist positive constants c_{11} and c_{12}, independent of h, such that*

$$c_{11}(A^{-1}u, u) \leq (RC_M^{-1}R^* u, u) \leq c_{12}(A^{-1}u, u) \qquad \forall u \in \mathrm{IR}^N .$$

Proof. The theorem is proved by using the argument of Theorem 3.2 and then that of Theorem 3.1. Indeed, at the first step, let us 'extend' the Dirichlet boundary condition from S_0^h to the boundary of the triangulation Π^h. To do it, we consider finite element space $H_h(G^h)$ and define

$$C_G^{-1} U^h = \sum_{l=0}^{J} \sum_{\text{supp}\, \Phi_i^{(l)} \subset G^h} (U^h, \Phi_i^{(l)})_{L_2(G^h)}\, \Phi_i^{(l)} \qquad \forall U^h \in H_h(G^h).$$

Then, according to Theorem 3.2, there exist positive constants c_{13}, c_{14}, independent of h, such that

$$c_{13} \|U^h\|^2_{H^1(G^h)} \leq (C_G U, U) \leq c_{14} \|U^h\|^2_{H^1(G^h)} \qquad \forall U^h \in H_h(G^h).$$

At the second step, define

$$R_{N,G}\colon\ H_h(G^h) \to H_h(Q^h)$$

as a restriction on Q^h from G^h:

$$(R_{N,G} U^h)(Z_{ij}) = U^h(Z_{ij}) \qquad \forall Z_{ij} \in Q^h .$$

Then, from Lemma 2.1 we get

$$c_{15}(A_Q^{-1} U, U) \leq (R_{N,G} C_G^{-1} R_{N,G}^* U, U) \leq c_{16}(A_Q^{-1} U, U) \qquad \forall U^h \in H_h(Q^h)$$

where c_{15}, c_{16} are independent of h. Using again the explicit form of $R_{N,G}$, we complete the proof of Theorem 3.3.

4 LOCALLY REFINED GRIDS

In this section we consider a triangulation Ω^h of the domain Ω

$$\Omega^h = \bigcup_{i=1}^{M} \tau_i$$

and assume Ω^h is regular but not quasi-uniform, i.e. there exists a constant s, independent of h, such that

$$\frac{r_i}{\rho_i} \leq s\,, \qquad i = 1, \ldots, M$$

where r_i and ρ_i are radii of circumscribed and inscribed circles for the triangle τ_i, respectively. It means that Ω^h can be locally refined. For this triangulation Ω^h, we define the space $H_h(\Omega^h)$ of real continuous functions which are linear on each triangle τ_i of Ω^h. For the sake of simplicity, we consider the Dirichlet boundary condition and assume that the functions from $H_h(\Omega^h)$ vanish at Γ^h.

If we introduce an uniform fictitious grid Q^h, then it is possible to modify the operators R and T from Section 2 for locally refined triangulation Ω^h, but the realization of a preconditioner will be expensive.

Let us embed the domain Ω in a square Π and start with a coarse uniform grid Π_0^h. We refine Π_0^h several times

$$\Pi_0^h, \Pi_1^h, \ldots$$

The grid Π_l^h consists of cells $D_{ij}^{(l)}$. Let Q_0^h denote the minimum figure that consists of cells $D_{ij}^{(0)}$ and contains Ω^h. Denote by I_0 a set of indices (i,j) such that

$$Q_0^h = \bigcup_{(i,j) \in I_0} D_{ij}^{(0)}$$

We define grids $Q_1^h, Q_2^h, \ldots$ in the following way. Denote by I_l a set of indices (i,j) such that the cell $D_{ij}^{(l)}$ contains more then one vertex of the triangulation Ω^h. We divide $D_{ij}^{(l)}$ and all neighbouring cells (which have at least one common node with $D_{ij}^{(l)}$) into four congruent sub cells by connecting the midpoints of the edges. Denote new cells by $D_{ij}^{(l+1)}$ and a resulting grid by $Q_{l+1}^h, l = 0, 1, \ldots$, which are the minimum figure that contains Ω^h. We stop this process when each cell contains no more than one vertex of Ω^h. Denote by Q_J^h the final grid.

Define a finite-element space $H_h(Q^h)$ as follows

$$H_h(Q^h) = \{ \sum_{\mathrm{supp}\,\Phi_k^{(0)} \subset Q_J^h} \alpha_k^{(0)} \Phi_k^{(0)} + \sum_{l=0}^{J-1} \sum_{(i,j)\in I_l} \sum_{\mathrm{supp}\,\Phi_k^{(l+1)} \cap D_{ij}^{(l)} \neq \emptyset} \alpha_k^{(l+1)} \Phi_k^{(l+1)} \mid \alpha_k^{(l)} \in \mathrm{I\!R} \}$$

we now define the projection operator R

$$R\colon\ H_h(Q^h) \to H_h(\Omega^h)$$

the extension operator T

$$T\colon\ H_h(\Omega^h) \to H_h(Q^h)$$

according to the definitions from Section 2.

Define a preconditioning operator in $H_h(Q_J^h)$ in the following way:

$$\begin{aligned} C_R^{-1} U^h &= \sum_{\mathrm{supp}\,\Phi_k^{(0)} \subset Q_J^h} (U^h, \phi_k^{(0)})_{L_2(Q_J^h)} \Phi_k^{(0)} \\ &+ \sum_{l=0}^{J-1} \sum_{(i,j)\in I_l} \sum_{\mathrm{supp}\,\Phi_k^{(l+1)} \cap D_{ij}^{(l)} \neq \emptyset} (U^h, \phi_k^{(l+1)})_{L_2(Q_J^h)} \Phi_k^{(l+1)} \end{aligned}$$

for any $U^h \in H_h(Q_J^h)$.

Theorem 4.1 *There exist positive constants c_{17} and c_{18}, independent of h, such that*

$$c_{17}(A^{-1}u, u) \le (RC_R^{-1}R^*u, u) \le c_{18}(A^{-1}u, u) \qquad \forall u \in \mathrm{I\!R}^N .$$

Proof. In this case, we again use the equivalence of H^1-norms of finite-element functions in the spaces $H_h(\Omega^h)$, $H_h(Q^h)$ and the difference counterparts of these norms and the multilevel technique.

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